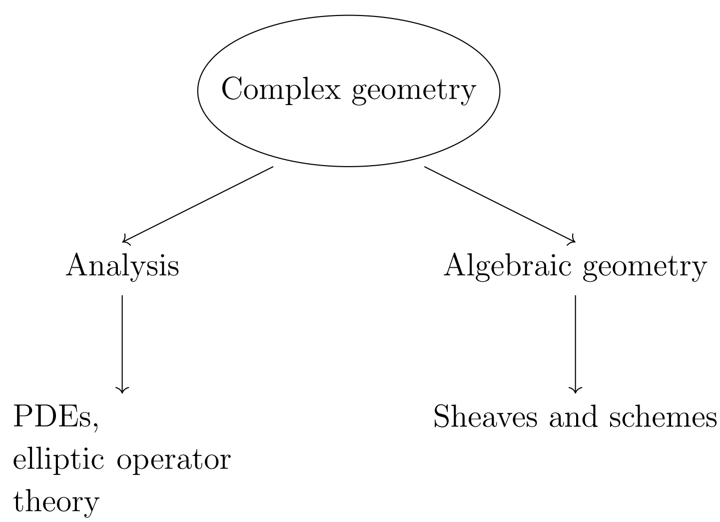
Just enough complex geometry to be dangerous

Abstract

There have been quite a few talks on both the analysis and algebra side about complex manifolds lately, as such this talk will give an introduction to the topic with a focus on the big ideas in play. To focus more on the big ideas, I will take a broad and shallow approach, avoiding most technicalities. I will also try to highlight the rich interplay between the analytic and algebraic aspects of the theory, with perhaps a bias towards the algebraic side.

1 Picture



Main claim: Complex geometry is interesting because there's a rich interplay between the two perspectives of algebra and analysis!

2 Complex manifolds

Central objects of study: Complex manifolds/complex analytic spaces
Question: What is a complex manifold?

This turns out to be a bit of a hard question with some subtlety!

Naïve answer: A complex manifold is a topological space covered by charts such that the transition functions $U \subset \mathbb{C}^n \to V \subset \mathbb{C}^n$ are holomorphic.

The problem is that this is a lot of information to check at once, and it turns out it's better to define it as a real manifold with some extra structure.

Better answer: A complex manifold is a real $2n$-dimensional manifold equipped with an "integrable almost complex structure."

Which begs the question: What does that mean?

2.1 Some technical nonsense

For those not interested, feel free to tune out, but there is a bit of subtlety here.

Definition 1. Let X be a real $2n$-dimensional manifold. An **almost complex structure** on X is an endomorphism

$$I : TX \to TX$$

such that $I^2 = -1$. A manifold equipped with an almost complex structure, (X, I), is called an **almost complex manifold**.

The reason this is called an almost complex structure is because this I isn't always induced from a complex manifold using that holomorphic transition function sense, and so it doesn't always give us an honest to goodness complex manifold. To fix this, we will need some extra work.

Given our tangent bundle TX on X, we can tensor with $\mathbb{C}$ to get $TX \otimes_{\mathbb{R}} \mathbb{C} = T_{\mathbb{C}}X$, and we can extend I to be an endomorphism on $T_{\mathbb{C}}X$. Since $I^2 = -1$, this means that I has eigenvalues $+i$ and $-i$, and we can break up $T_{\mathbb{C}}X$ into eigenspaces:

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$$

where $T^{1,0}X$ is the i eigenspace and $T^{0,1}X$ is the $-i$ eigenspace. We will call $T^{1,0}X$ the holomorphic part and $T^{0,1}X$ the antiholomorphic part. We can think of $T^{1,0}X$ as being (locally) spanned by

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$$

and the antiholomorphic part as being spanned by

$$\frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}.$$

We will call $T^{1,0}X = T_X$ the complex tangent bundle of X.

Definition 2. An almost complex structure I on X is called **integrable** if $[T_X, T_x] \subset T_X$.

One should think of this condition as what happens in the Frobenius Theorem, where here, the tangent vectors ∂_{z_i} integrate to give you holomorphic coordinates z_i.

TECHNICAL PART OVER

3 Kähler manifolds

Now the the whole world of complex manifolds is very scary, so we will focus for the rest of the talk on the best kind of manifold (I mean this without reservation): Kähler manifolds.

Definition 3. A complex manifold (X, I) is called **Kähler** if there exists a symplectic form ω on X such that

$$g(-,-) = \omega(-, I-)$$

is a symmetric nondegenerate bilinear form—i.e. a metric on X. ω is called the **Kähler form**.

Here symplectic form means an alternating form $\omega : TX \otimes TX \to TX$ that is nondegenerate, which means for any $v \in T_pX$ we have that $\omega(v, -)$ is an isomorphism $T_pX \to T_pX$, and closed, i.e. $d\omega = 0$.

Remark. There are many equivalent definitons of Kähler manifolds, which I won't get into.

Example 1. $\mathbb{C}^n$ with its usual symplectic form is a Kähler manifold. For those unfamiliar: $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ has real coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $z_i = x_i + iy_i$ and the symplectic form here is $\sum dy_i \wedge dx_i$.

Example 2. This is the most important example. $\mathbb{P}^n(\mathbb{C})$ has a Kähler form called the Fubini-Study form. It's given locally as

$$\omega = \frac{1}{2\pi i}\partial\overline{\partial}\log\left(\frac{1}{1+\sum_i |z_i|^2}\right),$$

where here $\partial, \overline{\partial}$ are the **Dolbeaut operators**:

$$\partial f = \sum_i \frac{\partial f}{\partial z_i} dz_i \quad \text{and} \quad \overline{\partial} f = \sum_i \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i.$$

The fact that $\mathbb{P}^n(\mathbb{C})$ is Kähler is actually very important! Let X be a smooth projective variety (over $\mathbb{C}$), i.e. $X \subset \mathbb{P}^N(\mathbb{C})$ is the vanishing locus of some homogeneous polynomials $\{F_i(x_0, \ldots, x_N)\}$, and we require the locus to be a complex manifold (smoothness condition). Now we can pullback along the inclusion $X \hookrightarrow \mathbb{P}^n(\mathbb{C})$ to give X the structure of a Kähler manifold, so we have that every smooth projective variety is Kähler! If you're an algebraic geometer, this makes you very happy.

Big question: When is a Kähler manifold projective?

We have one partial answer due to Kodaira. As setup, note that since ω, the Kähler form, is closed, we have that $[\omega] \in H^2(X, \mathbb{C})$.

Theorem 1 (Kodaira Embedding Theorem)**.** *Let X be a compact Kähler manifold with a Kähler form ω. If $[\omega] \in H^2(X, \mathbb{C})$ is an integral cohomology class, i.e. in this special case we can think of it as living in $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$, then X is projective.*

4 Just a bit of Hodge theory

When X is Kähler, the cohomology of X decomposes in a particularly nice way:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

This decomposition is called the **Hodge decomposition**. Here $H^{p,q}(X)$ are forms of the form

$$\alpha \, dz_{i_1} \wedge dz_{i_2} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge d\overline{z}_{j_2} \wedge \ldots \wedge d\overline{z}_{j_q},$$

i.e. forms with p dz's and q $d\overline{z}$'s. For the algebra pilled, there is also another description using cohomology:

$$H^{p,q}(X) \simeq H^q(X, \Omega_X^p),$$

where here Ω_X is the sheaf of (holomorphic) differential forms on X and $\Omega_X^p = \bigwedge^p \Omega_X$.

The main takeaway here is that these $H^{p,q}$'s form extra invariants of Kähler manifolds, which one would like if one is interested in classification questions. The numbers $h^{p,q} = \dim H^{p,q}(X)$ are called the Hodge numbers and they are often displayed in what's called the Hodge diamond:

$$\begin{array}{ccccccccc}
 & & & & h^{n,n} & & & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & \iddots & & & & \ddots & & \\
 & h^{n,1} & & & & & & h^{1,n} & \\
h^{n,0} & & & & & & & & h^{0,n} \\
 & h^{n-1,0} & & & & & & h^{0,n-1} & \\
 & & \ddots & & & & \iddots & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & & & h^{0,0} & & & &
\end{array}$$

Example 3. A big example of application of Hodge theory came from Clemens and Griffith ca. 1972. They used Hodge theory to show that cubic 3-folds, i.e. $X \subset \mathbb{P}^4(\mathbb{C})$ defined by the vanishing of a degree 3 homogeneous polynomial $F(x_0, x_1, x_2, x_3, x_4)$, are not bimeromorphic to $\mathbb{P}^3(\mathbb{C})$.

5 More explicit relations to algebraic geometry

I should justify why I said complex geometry is so closely relate to algebraic geometry.

Theorem 2 (Chow). *If X is a compact complex manifold with a closed embedding $X \hookrightarrow \mathbb{P}^N(\mathbb{C})$, then X is a projective variety.*

What this means then is that studying projective complex manifolds is really just studying algebraic geometry using these different tools! And in fact, one can actually say more in this direction.

5.1 Serre's GAGA

Suppose X is a complete variety over $\mathbb{C}$. What this means is that X is a reduced scheme of finite type (maybe also ask irreducible) over $\mathbb{C}$ such that the structure morphism $X \to \operatorname{Spec} \mathbb{C}$ is proper. The set $X(\mathbb{C}) = \operatorname{Hom}(\operatorname{Spec} \mathbb{C}, X)$ (whatever this means [really means the closed points of X]) has the structure of a compact complex analytic space. Here, complex analytic space means that locally, it looks like the vanishing locus of some holomorphic functions defined on an open subset of $\mathbb{C}^n$. We call this space X^{an} because we can give it the standard Hausdorff topology, which is very different from what one can get from the Zariski topology.

Now, a priori, X and X^{an} look like very different objects. But Serre's GAGA states roughly that

- Morphisms $X^{an} \to Y^{an}$ are the same as morphisms of schemes $X \to Y$.
- More importantly: Coherent analytic sheaves over X^{an} are the "same" as coherent sheaves over X, in the sense that the sheaves are the same, the morphisms of sheaves are the same, the exact sequences of sheaves are the same, and amazingly the cohomology of these sheaves are the same!

To really hammer home how crazy this statement is, just think about vector bundles:

In algebra land:	In complex geometry land:
In the Zariski topology, open sets are absolutely massive!	Small open sets are allowed, for example one can have ε-balls in the complex topology.
There are very few algebraic functions: they're all defined by polynomials, so there shouldn't be too many possible transition maps	There are seemingly more holomorphic functions allowed than just polynomials, so we seem to be able to have more transition functions

From this, we should really expect there to be a lot fewer algebraic vector bundles than holomorphic vector bundles. But actually, by GAGA, they are the same and they even have the same cohomology!