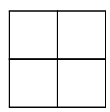
# Notes on quantum cohomology

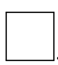
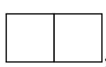
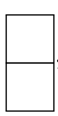
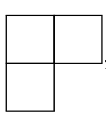
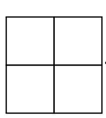
Xingyu Cheng

## 1 Cohomology ring of the Graßmannian

These notes are my attempts to understand quantum cohomology, specifically for that case of $Gr(r,n)$ the Grassmannian of rank $r$ subspaces of $\mathbb{C}^n$. So first we look at the regular cohomology ring of the Grassmannian $X = Gr(r,n)$. It is well known that the cohomology ring $H^*(X,\mathbb{Z})$ is generated by Schubert cells $\Omega_\lambda$ for $\lambda$ a partition of the square with $r \times (n-r)$ blocks, i.e. partitions are thought of as Young tableaux. For example, lets take the simplest case of $Gr(2,4)$. So here Schubert cells are parametrized by Young diagrams of the $2 \times 2$ box.



So here all the possible Schubert cells are

- $(1,0)$ ,
- $(2,0)$ ,
- $(1,1)$ ,
- $(2,1)$ ,
- $(2,2)$ .

These partitions $\lambda$ translate to $\Omega_\lambda$ in the following way. Let $F_\bullet = F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n$ be a flag on $\mathbb{C}^n$. The Schubert cell parametrized by $\lambda$ is defined to be

$$\Omega_\lambda(F_\bullet) = \{V \in Gr(r,n) \mid \dim(V \cap F_{n-r+i-\lambda_i}) \geq i, \forall 1 \leq i \leq r\}.$$

Alternatively note that each Schubert cell is defined by a path from the top right of the $r \times (n-r)$ rectangle to the bottom left. We can define $\Omega_\lambda(F_\bullet)$ to be subspaces $V \in Gr(r,n)$ such that $\dim(V \cap F_i)$ is greater than or equal to the row you are on after taking $i$ steps in the mentioned path. These two definitions are equivalent.

Now the Schubert calculus is determined basically by the Pieri and Giambelli formulas. The Giambelli formula gives us a presentation of the Schubert class $\Omega_\lambda$ as a determinant

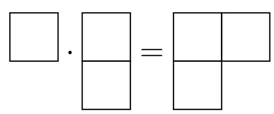
$$\Omega_\lambda = \det(\Omega_{\lambda_i+j-i})_{1\le i,j\le r}.$$

And then the Pieri formula states

$$\Omega_i \cdot \Omega_\lambda = \sum \Omega_\nu$$

where $\nu$ ranges over all partitions that can be obtained by adding $i$ boxes to the Young diagram of $\lambda$ with no two boxes being on the same column.

So now maybe lets do some examples with $Gr(2,4)$. We'll need to have $F_\bullet$ and $G_\bullet$ be two general flags, and let $H_\bullet$ be the flag you get by taking the intersection of the spaces of $F_\bullet$ and $G_\bullet$. For our example calculation, lets take the case of $\lambda = (1,1)$, then our Schubert cell is $\Omega_\lambda(F_\bullet) = \{V \mid \dim(V \cap F_{n-r+i-\lambda_i}) \ge i\} = \{V \mid \dim(V \cap F_2) \ge 1, \dim(V \cap F_3) \ge 2\}$. Lets look at what happens when we do $\Omega_1 \cdot \Omega_\lambda$. Diagrammatically our calculation looks like

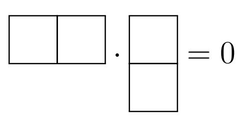


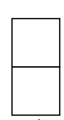
with no other options for summands on the right hand side. So we get that $\Omega_1 \cdot \Omega_\lambda = \Omega_{(2,1)}$. Note that $\Omega_1(G_\bullet) = \{V \mid \dim(V \cap G_2) \ge 1, \dim(V \cap G_4) \ge 2\}$, and so we have that

$$\begin{aligned}&\Omega_1(G_\bullet) \cap \Omega_\lambda(F_\bullet)\\ &\quad = \{\dim(V \cap G_2) \ge 1), \dim(V \cap G_4) \ge 2\} \cap \{\dim(V \cap (F_2) \ge 1), \dim(V \cap F_3) \ge 2\}\\ &\quad = \Omega_{(2,1)}(H_\bullet) = \{\dim(V \cap H_1) \ge 1, \dim(V \cap H_3) \ge 2\}.\end{aligned}$$

Now this calculation makes sense because two general planes $F_2$ and $G_2$ meet in a line, which we have here be $H_1$ and $G_4 = \mathbb{C}^4$ and so $H_3 = G_4 \cap F_3$ is a 3 dimensional subspace. Now, notice how there's a choice of subspace $H_3$ (it's either $F_3$ or $G_3$), but it doesn't matter in the end because any choice we make would still be in the same class in cohomology.

Next, lets do the calculation of $\Omega_2 \cdot \Omega_\lambda$ for the same $\lambda$ as above. So we have here



because there is not possible way for us to add two boxes to  without either going over bounds or putting two boxes on the same column. So lets try to interpret this result in terms of intersections then. Again, $F_\bullet, G_\bullet, H_\bullet$ general flags.

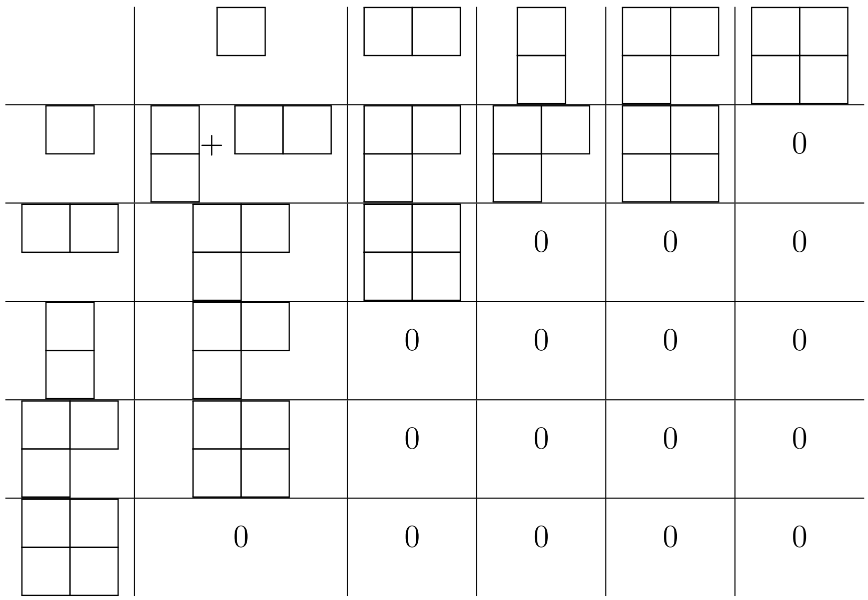
$$\Omega_2(F_\bullet) = \{\dim(V \cap F_1) \ge 1, \dim(V \cap F_4) \ge 2\}.$$

So looking at it in terms of intersection, we have that

$$\begin{aligned}
&\Omega_2(F_\bullet) \cap \Omega_{(1,1)}(G_\bullet) \\
&\quad = \{\dim(V \cap F_1) \geq 1, \dim(V \cap F_4) \geq 2\} \cap \{\dim(V \cap G_2) \geq 1, \dim(V \cap G_3) \geq 2\} \\
&\quad = \emptyset.
\end{aligned}$$

This makes sense because for general flags, we need to have that $F_1 \cap G_2 = 0$, so there can't be any subspaces $V \in Gr(2,4)$ that satisfies the first condition of both.

Alright, now lets try to draw the complete multiplication table.



So we get that the above becomes our multiplication table.

## 2 Quantum cohomology

So now that we have a basic handle on regular cohomology of Grassmannians (Graßmannian), we can move on to quantum cohomology. To start off, let us review some stuff about stable maps and Gromow-Witten numbers. The degree of a map $f : \mathbb{P}^1 \to \mathbb{P}^n$ is equal to the number of points in $f^{-1}(H)$, where $H$ is a general hyperplane in $\mathbb{P}^n$. The degree of a map $f : \mathbb{P}^1 \to Gr(r,n)$ is equal to the degree of the composition of $f$ with the Plücker embedding $Gr(r,n) \hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$ (with the map being of course $V \mapsto \bigwedge^r V$ which is a line). If we have $F_\bullet, G_\bullet, H_\bullet$ are three general flags, and $\lambda, \mu, \nu$ are three partitions that define Young taleaux in the $r \times (n-r)$ square, then these still give us the Schubert varieties $\Omega_\lambda(F_\bullet), \Omega_\mu(G_\bullet), \Omega_\nu(H_\bullet)$.

Now we consider the moduli space of stable maps $M = \overline{M_{0,3}}(d, Gr(r,n))$. This is the space of maps $f : \mathbb{P}^1 \to Gr(r,n)$ of degree $d$, satisfying some suitable stability conditions. Now define $\rho_i$ to be the $i$-th evaluation map $M \to Gr(r,n)$, with $\rho_i(f) = f(p_i)$. From here, we define the Gromow-Witten number by

$$\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = \int_M \rho_1^*(\Omega_\lambda) \cup \rho_2^*(\Omega_\mu) \cup \rho_3^*(\Omega_\nu).$$

Some remarks about this definition:

- This number is 0 if $|\lambda| + |\mu| + |\nu| \neq \dim M = \dim Gr(r,n) + \int_{Gr(r,n)} c_1(T_{Gr(r,n)}) = r(n-r) + nd$.
- This number counts the number of degree $d$ maps from $\mathbb{P}^1$ that sends $0, 1, \infty$ to the Schubert cells $\Omega_\lambda(F_\bullet)$, $\Omega_\mu(G_\bullet)$, and $\Omega_\nu(H_\bullet)$ respectively, up to automorphisms of $\mathbb{P}^1$. Note also that such degree $d$ maps can also be thought of as rational degree $d$ curves.
- This number is well defined, i.e. does not depend on choice of general flags $F_\bullet, G_\bullet, H_\bullet$.

The quantum cohomology ring $QH^*X = QH^*(X, \mathbb{Z})$ of $X$ is a $\mathbb{Z}[q]$-algebra which is isomorphic to $H^*X \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a $\mathbb{Z}[q]$ module. We still have Schubert classes $\sigma_\lambda = \Omega_\lambda \otimes 1$, and the multiplication is defined by

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu, d \geq 0} \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_d q^d \sigma_\nu$$

where $\nu^\vee$ is the dual partition to $\nu$, i.e. $\nu^\vee = ((n-r) - \nu_r, \ldots, (n-r) - \nu_1)$. It is a nontrivial fact that this actually defines an associative ring structure. Further, this ring has a graded structure with $\sigma_\lambda$ having degree $|\lambda|$ and $q$ having degree $n$.

There are quantum analogs of the Pieri and Giambelli formulas, which will tell us more about the structure of this quantum cohomology ring.

# 3 The span and kernel of a curve

Following Buch's treatment of quantum cohomology, the technical tools to use here are the span and kernel of a subvariety $Y$ of $X = Gr(r,n)$. The *span* of $Y$ is the smallest subspace of $\mathbb{C}^n$ that contains all the $r$-dimensional subspaces given by points of $Y$. The *kernel* of $Y$ is the largest subspace contained in all the $r$-dimensional subspaces given by points of $Y$.

**Lemma 1.** *Let $C$ be a rational curve of degree $d$ in $X$. Then the span of $C$ has dimension at most $r + d$ and the kernel of $C$ has dimension at least $r - d$.*

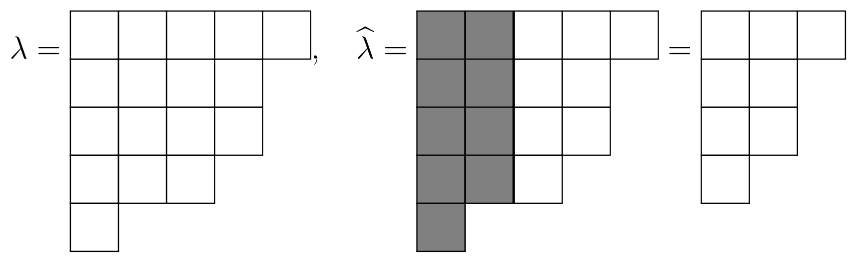
*Proof.* So let $C$ be the curve given by the image of $f : \mathbb{P}^1 \to Gr(r,n)$, with $\deg f = d$. Let $S \subset \mathcal{O}_X^n$ be the tautological/universal bundle on $X$. Then we know that $f^*S = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(-a_i)$ with each $a_i \geq 0$ and $\sum a_i = d$. Now $f$ is given by the inclusion $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(-a_i) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^n$ (since the inclusion we can think of as just locally precomposing by $f$). This inclusion works by sending a point $p \in \mathbb{P}^1$ to the fiber over $p$ of the image of this map.

Now if $[s,t]$ are homogeneous coordinates on $\mathbb{P}^1$, then $\Gamma(\mathcal{O}_{\mathbb{P}^1}(a_i))$ is generated by sections of the form $s^j t^{a_i - j}$, $0 \leq j \leq a_i$. Now each map $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(-a_i) \to \mathcal{O}_{\mathbb{P}^1}^n$ is given by

$$\sum_{j=0}^{a_i} \alpha_j s^{-j} t^{j-a_i} \mapsto \sum_{j=0}^{a_i} \alpha_j v_j^i,$$

with $v_j^i$ being some vector in $\mathbb{C}^n$ (and of course repeat the above for all $i$). So then we get that every vector is in the span of $v_j^i$ for all $i, j$. For each $j$ there are $1 + a_i$ vectors, and so we have a total of $\sum_{i=1}^r (1 + a_i) = r + d$. So we get that the span of $Y$ has dimension at most $r + d$. For the dimension of the kernel, note that we need $\sum a_i = d$, but if we don't have at least $r - d$ of the $a_i = 0$, say we have $m < r - d$ zero, then we get that since each $a_i \geq 0$, $r - m \leq \sum_{i=1}^r a_i = d$, and so rearranging would get us that $r - d \leq m$, which contradicts our assumption in the first place. □

If $\lambda$ is a partition, $d \geq 0$ an integer, then denote $\widehat{\lambda}$ to be the partition obtained by removing the left $d$ columns from the Young tableau of $\lambda$. I.e. we have that $\widehat{\lambda}_i = \max\{\lambda_i - d, 0\}$. In pictures, this should be something like the following.



where the grayed out part means the $d$ columns that we removed.

**Lemma 2.** *Let $C \subset X$ be a rational curve of degree $d \leq n - r$ and let $W \subset \mathbb{C}^n$ be an $r + d$ dimensional subspace containing the span of $C$. If $\lambda$ is a partition such that $C \cap \Omega_\lambda(F_\bullet) \neq \emptyset$, then $W$ belongs to the Schubert variety $\Omega_{\widehat{\lambda}}(F_\bullet)$ in $Gr(r + d, n)$.*

*Proof.* Let $V \in C \cap \Omega_\lambda(F_\bullet)$. $V \subset W$ because $V$ is in $C$. But now, since $V$ is in $\Omega_\lambda(F_\bullet)$, this implies that $\dim(V \cap F_{n-r+i-\lambda_i}) \geq i$, and consequently, $\dim(W \cap F_{n-r+i-\lambda_i}) \geq i$ as well. But, this is really the same as saying that $W \in \Omega_{\widehat{\lambda}}(F_\bullet)$, since this is the conditions that $\dim(W \cap F_{n-r+d+i-\max(\lambda_i - d, 0)}) \geq i$, and the $d$'s cancel out. □

We can say something similar about the kernel as well. If $S \subset \mathbb{C}^n$ is an $r - d$ dimensional subspace contained in the kernel of $C$, then $S \in \Omega_{\overline{\lambda}}(F_\bullet) \subset Gr(r - d, n)$, where $\overline{\lambda} = (\lambda_{d+1}, \ldots, \lambda_r)$ is the result of removing the first (top) $d$ rows. So we get that the condition that a curve meets a given set of Schubert varieties in $X$ implies that the intersections of related Schubert varieties in $Gr(r + d, n)$ and $Gr(r - d, n)$ are not empty, which is then a statement about the usual cohomology of these spaces. So in some sense, we actually reduced our question to the usual cohomology. This idea is actually sufficient to compute Gromov-Witten invariants in many importantt cases!

Now lets move on to some of the quantum versions of the Pieri-Giambelli formulas.

# 4 Quantum Pieri formula

**Theorem 1.** *Let $\lambda$ be a partition contained in the $r \times (n - r)$ rectangle. Let $p \leq n - r$, then*

$$\sigma_p \cdot \sigma_\lambda = \sum \sigma_\mu + q \sum \sigma_\nu$$

*where $\mu$ ranges over all partitions such that $|\mu| = |\lambda| + p$ and*

$$n - r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \geq \mu_r \geq \lambda_r,$$

*and $\nu$ ranges over all partitions with $|\nu| = |\lambda| + p - n$ and*

$$\lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \ldots \geq \lambda_r - 1 \geq \nu_r \geq 0.$$

Some remarks before the proof that will be of interest later. Let $\ell(\lambda)$ be the number of nonzero parts of $\lambda$. Then we have that

$$\sum \sigma_\nu \neq 0$$

if and only if $\ell(\lambda) = r$, since if not then we get some $\lambda_i - 1 = -1$ which is of course not greater than or equal to zero; this is the same as saying that we want each $\lambda_i \geq 1$.

*Proof.* So we look first at the $q$ degree 0 term. This sum is gotten by the classical Pieri formula. The classical case is equivalent to the following statement:

If $\alpha, \beta$ are partitions such that $|\alpha| + |\beta| + p = r(n - r)$, then

$$\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_0 = \begin{cases} 1 & \text{if } \alpha_i + \beta_j \geq n - r \text{ for } i + j = r \text{ and } \alpha_i + \beta_j \leq n - r \text{ for } i + j = r + 1, \\ 0 & \text{else.} \end{cases}$$

So first why is the first sum equivalent to the above statement? Well first, clearly this is because the above sum is the case when $d = 0$. Which again is the sum over $\mu$ for $n - r \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r$. Remember in the definition of quantum product, we have that

$$\sigma_p * \sigma_\lambda = \sum_\mu \langle \Omega_p, \Omega_\lambda, \Omega_{\mu^\vee} \rangle_0 \sigma_\mu + \sum_{\nu, d \geq 0} \langle \Omega_p, \Omega_\lambda, \Omega_\nu \rangle_d q^d \sigma_\nu.$$

So basically, what we need is to have $\langle \Omega_p, \Omega_\lambda, \Omega_{\mu^\vee} \rangle_0 = 1$ precisely when $n - r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \ldots \geq \mu_r \geq \lambda_r$. First, lets prove that we need $|\mu|$ to be what it is. We want $p + |\lambda| + |\mu^\vee| = r(n - r)$. Now $|\mu^\vee| = r(n - r) - |\mu|$, so rearranging gives us exactly what we want. Ok, next we have that

$$\mu_i^\vee = n - r - \mu_{r-i+1}.$$

Now going back to the conditions on $\alpha, \beta$, we look at $\lambda_i + \mu_j^\vee$ for $i + j = r$. In this case, $j = r - i$, so then we get the condition

$$\lambda_i - \mu_{i+1} + (n - r) \geq (n - r)$$

which is equivalent to $\lambda_i \geq \mu_{i+1}$. Now the condition $\lambda_i + \mu_j^\vee \geq n - r$ for $i + j = r + 1$ is equivalent to

$$\lambda_i - \mu_{r-j+1} \leq 0.$$

$j = r + 1 - i$, so we get that $\lambda_i - \mu_i \leq 0$, so $\lambda_i \leq \mu_i$. So indeed this is equivalent to the condition $n - r \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r$.

Now look at the rank $d \geq 1$ case. Suppose now that $|\alpha| + |\beta| + p = r(n-r) + nd$, i.e. the dimension counts match. Let $C$ be a rational curve of degree $d$ on $X$ meeting each $\Omega_\alpha(F_\bullet), \Omega_\beta(G_\bullet), \Omega_p(H_\bullet)$ for general flags $F_\bullet, G_\bullet, H_\bullet$. Let $W \subset \mathbb{C}^n$ be a general subspace of dimension $r+d$ containing the span of $C$ (so this is where we're going to reduce to regular Schubert calculus on $Gr(r+d, n)$). Now $W \in Gr(r+d, n)$ lies in the intersection

$$\Omega_{\widehat{\alpha}}(F_\bullet) \cap \Omega_{\widehat{\beta}}(G_\bullet) \cap \Omega_{\widehat{p}}(H_\bullet)$$

where $\widehat{\alpha}, \widehat{\beta}$ is the partition you get after removing the left $d$ rows, and $\widehat{p} = \max(p-d, 0)$. Now, $F_\bullet, G_\bullet, H_\bullet$ general implies that

$$|\widehat{\alpha}| + |\widehat{\beta}| + |\widehat{p}| \leq (r+d)(n-r-d) = \dim Gr(r+d, n),$$

i.e. they don't intersect trivially. We also have that

$$|\widehat{\alpha}| + |\widehat{\beta}| + |\widehat{p}| \geq |\alpha| + |\beta| - 2rd + p - d = (r+d)(n-r-d) + d^2 - d.$$

This is because $|\widehat{\alpha}| = \sum^r \max(\alpha_i - d, 0) \geq |\alpha| - rd$, etc. Now since $d \geq 1$, this means that $d^2 - d \leq 0$, and hence

$$(r+d)(n-r-d) + d^2 - d \leq (r+d)(n-r-d).$$

So the only case when the Gromow-Witten number $\langle \Omega_{\widehat{p}}, \Omega_{\widehat{\alpha}}, \Omega_{\widehat{\beta}} \rangle_d \neq 0$ is when $d = 1$ in this case. $d = 1$ implies that we need $\ell(\alpha) = \ell(\beta) = r$, since we have that

$$\begin{aligned} |\widehat{\alpha}| + |\widehat{\beta}| + |\widehat{p}| &= (r+1)(n-r-1) \\ &= |\alpha| + |\beta| + p - 2r - 1 \\ &= (|\alpha| - r) + (|\beta| - r) + (p-1), \end{aligned}$$

so we need that each coefficient needs to be nonnegative.

So we get then that the quantum Pieri formula becomes equivalent to the case if $|\alpha| + |\beta| + p = r(n-r) + nd$, then

$$\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 = \begin{cases} 1 & \text{if } \alpha_i + \beta_j \geq k+1 \text{ for } i+j = r+1 \\ & \text{and } \alpha_i + \beta_j \leq k+1 \text{ for } i+j = r+2, \\ 0 & \text{else.} \end{cases}$$

Recasting this in terms of our partitions $\lambda, p, \nu$, we get that we need

$$\langle \Omega_\lambda, \Omega_p, \Omega_{\nu^\vee} \rangle_1 = 1.$$

Well, we need $\nu$ to be such that $|\nu^\vee| = r(n-r) - |\nu|$, and

$$|\nu^\vee| + |\lambda| + p = r(n-r) + n$$

which implies that $|\nu| = |\lambda| + p - n$. So this is how we get the dimension count. Next, we want to see how to recover our system of inequalities. So recall that we need

$$\lambda_1 - 1 \geq \mu_1 \geq \lambda_2 - 1 \geq \mu_2 \geq \ldots \geq \lambda_r - 1 \geq \mu_r \geq 0.$$

We need to check whether this condition is equivalent to $\lambda_i + \nu_j^\vee \geq k+1$ for $i + j = r+1$ and $\lambda_i + \nu_j^\vee \leq k+1$ for $i+j = r+2$.

So lets check the conditions: First for $i + j = r+1$, we get that

$$\lambda_i + \nu_j^\vee = \lambda_i - \nu_i + n - r \geq n - r + 1,$$

so rearranging gives us that $\lambda_i - 1 \geq \nu_i$. Next for $i + j = r + 2$, we get that

$$\lambda_i + \nu_j^\vee = \lambda_i - \nu_{i-1} + n - r \leq n - r + 1,$$

so rearranging here gives us that $\lambda_i - 1 \leq \nu_{i-1}$ (e.g. $\nu_1 \geq \lambda_2 - 1$). So indeed this is equivalent to the system of inequalities that we want.

So we get that

$$\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 = 0 \iff \langle \Omega_{\widehat{\alpha}}, \Omega_{\widehat{\beta}}, \Omega_{\widehat{p}} \rangle_0 = 0.$$

Here, the latter equality is just the classic Pieri rule. So we just reduced the quantum problem back to the classical case.

Now, if $\langle \Omega_{\widehat{\alpha}}, \Omega_{\widehat{\beta}}, \Omega_{\widehat{p}} \rangle_0 = 0$, then there does not exist a $W$, so actually $C$ can't exist. If $\langle \Omega_{\widehat{\alpha}}, \Omega_{\widehat{\beta}}, \Omega_{\widehat{p}} \rangle_0 = 1$, then there's only one $W \in Gr(r+1, n)$ that's contained in

$$\Omega_{\widehat{\alpha}}(F_\bullet) \cap \Omega_{\widehat{\beta}}(G_\bullet) \cap \Omega_{\widehat{p}}(H_\bullet)$$

(the intersection has full codimtension so must be a collection of points). Since the flags are general, $W$ must lie in the interior of each Schubert variety, so in particular we have that

$$V_1 = W \cap F_{(n-r)-\alpha_r+r} = W \cap F_{n-\alpha_r}$$

and

$$V_2 = W \cap G_{(n-r)-\beta_r+r} = W \cap F_{n-\beta_r}$$

all have dimension $r$. Also we have that $V_1 \in \Omega_\alpha(F_\bullet)$ and $V_2 \in \Omega_\beta(G_\bullet)$. Now we have that $\Omega_\alpha(F_\bullet) \cap \Omega_\beta(G_\bullet) = \emptyset$ (I think because of some shenanigans with regular Pieri rule, not completely sure), this implies that $V_1 \neq V_2$, so then we have that $\dim V_1 \cap V_2 = r - 1$ (the intersection increases the codimension by 1, so cuts down the dimension by 1). This implies that the only rational curve of degree 1 in $X$ that meets $\Omega_\alpha, \Omega_\beta, \Omega_p$ is the line $\mathbb{P}(W/S)$ of $r$-dimensional subspaces between $S$ and $W$.

Now lets check that the regular Pieri rule actually holds like we claim it does. So first checking the product

$$\Omega_p \cdot \Omega_\lambda = \sum_\mu \langle \Omega_p, \Omega_\lambda, \Omega_{\mu^\vee} \rangle_0 \Omega_m u = \sum \Omega_\mu,$$

with now $\mu$ being obtained in the regular way (i.e. adding $p$ boxes to $\lambda$ so that no two box ends up in the same column. But now this implies that we get every $\mu$ such that

$$n - r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \geq \mu_r \geq \lambda_r.$$

And we saw that the 2nd condition basically follows from the first by something like induction. $\square$

# 5 Quantum Giambelli

The quantum Giambelli is actually just the regular Giambelli formula with no modification. We take the convention that $\sigma_i = 0$ for $i < 0$ and $n - r < i < n$.

**Theorem 2** (Quantum Giambelli). $\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \leq i,j \leq r}$.

*Proof.* This proof just boils down to showing that the quantum product of Schubert cells with only one nonzero row is the same as the regular product. I.e. we prove first that if $a_1, \ldots, a_r$ are numbers between 0 and $n - r$, then

$$\sigma_{a_1} \cdot \ldots \cdot \sigma_{a_r} = (\Omega_{a_1} \cdot \ldots \cdot \Omega_{a_r}) \otimes 1.$$

We do this by induction. We prove first that if $\ell(\lambda) < r$, then $\sigma_i \cdot \sigma_\lambda$ involves no $q$ terms and no partitions of length greater than $\ell(\lambda) + 1$. So why is this? Well by formula we have that

$$\sigma_i \cdot \sigma_\lambda = \sum \sigma_\mu + q \sum \sigma_\nu,$$

where $\nu$ is such that $\lambda_1 - 1 \geq \nu_1 \geq \ldots \geq \lambda_r - 1 \geq \nu_r \geq 0$, $|\nu| = |\lambda| + i - n$. But since $\ell(\lambda) < r$, no such $\nu$ can exist, since no numbers are greater than or equal to 0 and less than or equal to $-1$. And in the above equation, the $\mu$'s are gotten by just doing the regular Schubert product, and in that product it is clear that we can't add more than 1 row to the Young tableau because then we would be adding more than one square to the same column. So indeed the claim holds. $\square$

Well actually a stronger claim is possible. If $\lambda, \mu$ are partitions contained in the $r \times (n-r)$ rectangle such that $\ell(\lambda) + \ell(\mu) \leq r$, then $\sigma_\lambda \cdot \sigma_\mu = (\Omega_\lambda \cdot \Omega_\mu) \otimes 1$.

*Proof.* If $d \geq 1$ and $\nu$ is such that $|\lambda| + |\mu| + |\nu| = r(n - r) + nd$ then any intersection

$$\Omega_{\widehat{\lambda}}(F_\bullet) \cap \Omega_{\widehat{\mu}}(G_\bullet) \cap \Omega_{\widehat{\nu}}(H_\bullet)$$

of general Schubert varieties in $Gr(r + d, n)$ must empty since

$$\begin{aligned} |\widehat{\lambda}| + |\widehat{\mu}| + |\widehat{\nu}| &\geq |\lambda| + |\mu| + |\nu| - 2dr \\ &= r(n - r) + nd - 2dr \\ &= r(n - r) + (n - r + r)d - 2dr \\ &= (r + d)(n - r) - dr \\ &> (r + d)(n - r - d). \end{aligned}$$

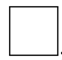
So the codimension of the intersection is too high, and hence must be empty. Thus we get that

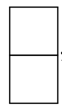
$$\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = 0.$$

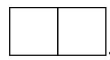
□

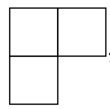
# 6 Some easy example computations

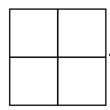
Here we do the case of $Gr(2,4)$. The possible partitions $\lambda$ are now

$\lambda = (1,0)$ ,

$(1,1)$ ,

$(2,0)$ ,

$(2,1)$ ,

$(2,2)$ .

Lets try an easy example of $\sigma_{(2,1)} \cdot \sigma_2 = \sum \sigma_\mu + q \sum \sigma_\nu$. First we look at $\mu$. We need

$$n - r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2, \quad \text{i.e.}$$

$$2 \geq \mu_1 \geq 2 \geq \mu_2 \geq 1$$

with the further restriction that $|\mu| = |\lambda| + p = 3 + 2 = 5$. This is just not possible, so there are no $q$-degree 0 terms. Next look at $\nu$. Here we need $|\nu| = |\lambda| + p - n = 5 - 4 = 1$. The system of inequalities we have is

$$\lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \nu_2 \geq 0, \quad \text{i.e.}$$

$$1 \geq \nu_1 \geq 0 \geq \nu_2 \geq 0.$$

The only possible $\nu$ in this case would be $(1,0)$. So we find that

$$\sigma_{(2,1)} \cdot \sigma_2 = q\sigma_1.$$

So here we already get something interesting because the regular Schubert calculus would've given us 0.

Next, try $\sigma_{(2,1)} \cdot \sigma_1 = \sum \sigma_\mu + q \sum \sigma_\nu$. Again first check $\mu$. We need $|\mu| = |\lambda| + p = 3 + 1 = 4$, and our system of inequalities is

$$n - r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2, \quad \text{i.e.}$$

$$2 \geq \mu_1 \geq 2 \geq \mu_2 \geq 1.$$

So the only option that fits these criteria is $\mu = (2,2)$. Next we check the $\nu$ conditions. We have that $|\nu| = |\lambda| + p - n = 4 - 4 = 0$, so the only $\nu$ here is 0. So we have that

$$\sigma_{(2,1)} \cdot \sigma_1 = \sigma_{(2,2)} + q\sigma_0.$$

Now what is $\sigma_0$? We have that $\Omega_0(F_\bullet) = \{\dim(V \cap F_{2-0+1}) = \dim(V \cap F_3) \geq 1, \dim(V \cap F_{2-0+2}) = \dim(V \cap F_4) \geq 2\} = Gr(2,4)$. Alternatively, one could've noticed that $|0| = \operatorname{codim} \Omega_0 = 0$. Actually, $\sigma_0 = 1$, so the above product is actually

$$\sigma_{(2,1)} \cdot \sigma_1 = \sigma_{(2,2)} + q.$$

Next, lets do one where we need the Giambelli rule as well. The only such case is really $\sigma_{(1,1)} \cdot \sigma_{(2,1)}$. Using the Giambelli rule, we get that

$$\begin{aligned}
\sigma_{(1,1)} &= \det(\sigma_{\lambda_i - j + i})_{1 \leq i,j \leq r=2} \\
&= \det \begin{pmatrix} \sigma_{1-1+1} & \sigma_{1-2+1} \\ \sigma_{1-1+2} & \sigma_{1-2+2} \end{pmatrix} = \det \begin{pmatrix} \sigma_1 & \sigma_0 \\ \sigma_2 & \sigma_1 \end{pmatrix} \\
&= \sigma_1 \sigma_1 - \sigma_2.
\end{aligned}$$

So then our product becomes

$$\sigma_1 \sigma_1 \sigma_{(2,1)} - \sigma_2 \sigma_{(2,1)}.$$

Well, we just computed $\sigma_2 \sigma_{(2,1)} = q\sigma_1$. We also computed $\sigma_1 \sigma_{(2,1)} = \sigma_{(2,2)} + q$, so we need to compute next

$$\sigma_1(\sigma_{(2,2)} + q).$$

$\sigma_1 \cdot \sigma_{(2,2)} = \sum \sigma_\mu + q \sum \sigma_\nu$. We know that no $\mu$'s are possible because of regular computations in cohomology. So now we just need to check the $\nu$ conditions. We have that $|\nu| = |\lambda| + p - n = 4 + 1 - 4 = 1$, and

$$1 \geq \nu_1 \geq \nu_2 \geq 0.$$

So the only possible $\nu$ is $(1,0)$. So we see that $\sigma_1 \sigma_{(2,2)} = q\sigma_1$. Putting everything together then we see that the final product is

$$(q\sigma_1 + q\sigma_1) - q\sigma_1 = q\sigma_1.$$

Interesting computation.

# 7 Converting from subsets to partitions

Next the question is how do we convert from subsets $I \subset [n]$ to partitions $\lambda$. Well, we have that for a flag $F_\bullet$, we have that

$$\Omega_\lambda(F_\bullet) = \{V \mid \dim(V \cap F_{n-r-\lambda_i+i}) \geq i\},$$

while for $I = \{i_1 \leq i_2 \leq \ldots \leq i_t\}$

$$\Omega_I(F_\bullet) = \{V \mid \dim(V \cap F_{i_a}) \geq a, i_a \in I\}.$$

So from here, it seems that the formula is clear. We set $a = i$ and check that

$$n - r - \lambda_a + a = i_a,$$

so actually we get that $n - r - i_a + a = \lambda_a$, so just do the same thing. So the conversion isn't bad at all.

# 8 Relating to inequalities for unitary product problem

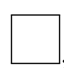
So then the next step is to relate these quantum cohomology calculations to the inequalities determining the unitary product problem.

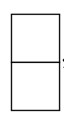
## 8.1 Rank 4 case

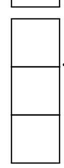
So lets say we have $V$ unitary-bundle of rank 4. We want to look at subbundles. So let $E \subset V$ be a subbundle of rank $r$ and degree $d$. These subbundles are of course equivalent to a map $f : \mathbb{P}^1 \to Gr(r,n)$. From here we have to break down the problem into various subcases.

**1** $\operatorname{rank} E = 3$

This is the case when we're looking at maps $f : \mathbb{P}^1 \to Gr(3,4) \simeq \mathbb{P}^3$. I guess, we should do the quantum cohomology table here as well. This is the case of an $3 \times 1$ box. So our partitions will be

$\lambda = (1,0,0)$ ,

$(1,1,0)$ ,

$(1,1,1)$ .

Alright, time to write down the quantum product table like before.

| | $\sigma_{(0,0,0)} = 1$ | $\sigma_{(1,0,0)}$ | $\sigma_{(1,1,0)}$ | $\sigma_{(1,1,1)}$ |
|---|---|---|---|---|
| $\sigma_{(0,0,0)} = 1$ | $1$ | $\sigma_{(1,0,0)}$ | $\sigma_{(1,1,0)}$ | $\sigma_{(1,1,1)}$ |
| $\sigma_{(1,0,0)}$ | $\sigma_{(1,0,0)}$ | $\sigma_{(1,1,0)}$ | $\sigma_{(1,1,1)}$ | $q$ |
| $\sigma_{(1,1,0)}$ | $\sigma_{(1,1,0)}$ | $\sigma_{(1,1,1)}$ | $q$ | $q\sigma_{(1,0,0)}$ |
| $\sigma_{(1,1,1)}$ | $\sigma_{(1,1,1)}$ | $q$ | $q\sigma_{(1,0,0)}$ | $q\sigma_{(1,1,0)}$ |

Now, lets write down the conversion to subsets. The conversion formula is $n-r-\lambda_i+i = I_i$, which translates to $1-\lambda_i+i = I_i$.

$$\begin{aligned}(0,0,0) &\rightsquigarrow \{2,3,4\},\\ (1,0,0) &\rightsquigarrow \{1,3,4\},\\ (1,1,0) &\rightsquigarrow \{1,2,4\},\\ (1,1,1) &\rightsquigarrow \{1,2,3\}.\end{aligned}$$

And then the dual conversion is $I_i^\vee = i + \lambda_{r-i+1} = i + \lambda_{3-i+1}$.

$$\begin{aligned}(0,0,0)^\vee &\rightsquigarrow \{1,2,3\},\\ (1,0,0)^\vee &\rightsquigarrow \{1,2,4\},\\ (1,1,0)^\vee &\rightsquigarrow \{1,3,4\},\\ (1,1,1)^\vee &\rightsquigarrow \{2,3,4\}.\end{aligned}$$

So from here in the rank 3 subbundle case, we can write down some of the inequalities from the above information. For a degree 1 inequality, we can look at $\sigma_{(1,1,1)} \cdot \sigma_{(1,1,0)} = q\sigma_{(1,1,0)}$. This translates to

$$\langle \sigma_{(1,1,1)}(F_\bullet), \sigma_{(1,1,0)}(G_\bullet), \sigma_{(1,1,0)^\vee}(H_\bullet)\rangle_1 = 1.$$

In the subset notation, this translates to

$$\langle \sigma_{\{1,2,3\}}(F_\bullet), \sigma_{\{1,2,4\}}(G_\bullet), \sigma_{\{1,3,4\}}(H_\bullet)\rangle_1 = 1.$$

This then translates to the inequality

$$a_4 + a_3 + a_2 + b_4 + b_3 + b_1 + c_4 + c_2 + c_1 \leq 1.$$

For a degree 0 inequality, we can use $\sigma_{(1,0,0)} \cdot \sigma_{(1,1,0)} = \sigma_{(1,1,1)}$. This translates to

$$\langle \Omega_{(1,0,0)}(F_\bullet), \Omega_{(1,1,0)}(G_\bullet), \Omega_{(1,1,1)^\vee}(H_\bullet)\rangle_0 = 1.$$

This then translates to

$$\langle \Omega_{\{1,3,4\}}(F_\bullet), \Omega_{\{1,2,4\}}(G_\bullet), \Omega_{\{2,3,4\}}(H_\bullet)\rangle_0 = 1.$$

This gives us the inequality

$$a_4 + a_2 + a_1 + b_4 + b_3 + b_1 + c_3 + c_2 + c_1 \leq 0.$$

## 2 rank $E = 2$

This case is the interesting one, because now we are looking at maps $f : \mathbb{P}^1 \to Gr(2,4)$, which is not a projective space. Here, we do actually need to use the Schubert calculus. So maybe we should just write out all the products in $QH^*Gr(2,4)$. So lets write down the table. We already know some of the products.

| | $\sigma_{(0,0)} = 1$ | $\sigma_{(1,0)}$ | $\sigma_{(2,0)}$ | $\sigma_{(1,1)}$ | $\sigma_{(2,1)}$ | $\sigma_{(2,2)}$ |
|---|---|---|---|---|---|---|
| $\sigma_{(0,0)} = 1$ | 1 | $\sigma_{(1,0)}$ | $\sigma_{(2,0)}$ | $\sigma_{(1,1)}$ | $\sigma_{(2,1)}$ | $\sigma_{(2,2)}$ |
| $\sigma_{(1,0)}$ | $\sigma_{(1,0)}$ | $\sigma_{(2,0)} + \sigma_{(1,1)}$ | $\sigma_{(2,1)}$ | $\sigma_{(2,1)}$ | $\sigma_{(2,2)} + q$ | $q\sigma_{(1,0)}$ |
| $\sigma_{(2,0)}$ | $\sigma_{(2,0)}$ | $\sigma_{(2,1)}$ | $\sigma_{(2,2)}$ | $q$ | $q\sigma_{(1,0)}$ | $q\sigma_{(1,1)}$ |
| $\sigma_{(1,1)}$ | $\sigma_{(1,1)}$ | $\sigma_{(2,1)}$ | $q$ | $\sigma_{(2,2)}$ | $q\sigma_{(1,0)}$ | $q\sigma_{(2,0)}$ |
| $\sigma_{(2,1)}$ | $\sigma_{(2,1)}$ | $\sigma_{(2,2)} + q$ | $q\sigma_{(1,0)}$ | $q\sigma_{(1,0)}$ | $q(\sigma_{(2,0)} + \sigma_{(1,1)})$ | $q\sigma_{(2,1)}$ |
| $\sigma_{(2,2)}$ | $\sigma_{(2,2)}$ | $q\sigma_{(1,0)}$ | $q\sigma_{(1,1)}$ | $q\sigma_{(2,0)}$ | $q\sigma_{(2,1)}$ | $q^2$ |

Next up, we want to convert the partitions into subsets $I \subset [n]$ each of cardinality 2. Recall from the above that the conversion is given by $I_i = n - r - \lambda_i + i$ So writing them down, we get that

$$\begin{aligned}
(0,0) &\rightsquigarrow \{3,4\}, \\
(1,0) &\rightsquigarrow \{2,4\}, \\
(2,0) &\rightsquigarrow \{1,4\}, \\
(1,1) &\rightsquigarrow \{2,3\}, \\
(2,1) &\rightsquigarrow \{1,3\}, \\
(2,2) &\rightsquigarrow \{1,2\}.
\end{aligned}$$

Next up, we also want to understand what the dual subsets should look like. Writing them down, we get the following table.

$$\begin{aligned}
(0,0)^\vee &\rightsquigarrow \{1,2\}, \\
(1,0)^\vee &\rightsquigarrow \{1,3\}, \\
(2,0)^\vee &\rightsquigarrow \{1,4\}, \\
(1,1)^\vee &\rightsquigarrow \{2,3\}, \\
(2,1)^\vee &\rightsquigarrow \{2,4\}, \\
(2,2)^\vee &\rightsquigarrow \{3,4\}.
\end{aligned}$$

Alright, now from here we should be able to just write the inequalities down, which sounds incredibly tedious. But there is a good way to write them all down from just looking at the table. There are too many to write completely down though. Maybe lets just write down some of the interesting ones. From $\sigma_{(2,2)} \cdot \sigma_{(2,2)} = q^2\sigma_{(0,0)}$, we get that the Gromow-Witten number

$$\langle \Omega_{\{1,2\}}(F_\bullet), \Omega_{\{1,2\}}(G_\bullet), \Omega_{\{1,2\}}(H_\bullet)\rangle_2 = 1,$$

which would imply the inequality

$$a_4 + a_3 + b_4 + b_3 + c_4 + c_3 \leq 2.$$

From $\sigma_{(2,0)} \cdot \sigma_{(2,1)} = q\sigma_{(1,0)}$, we get that

$$\langle \Omega_{\{1,4\}}(F_\bullet), \Omega_{1,3}(G_\bullet), \Omega_{1,3}(H_\bullet)\rangle_1 = 1,$$

which gives us the inequality

$$a_1 + a_4 + b_2 + b_4 + c_2 + c_4 \leq 1.$$

For a degree 0 inequality, we have that $\sigma_{(1,0)} \cdot \sigma_{(1,1)} = \sigma_{(2,1)}$, which would give us that

$$\langle \Omega_{\{2,4\}}(F_\bullet), \Omega_{\{2,3\}}(G_\bullet), \Omega_{\{2,4\}}(H_\bullet) \rangle_0 = 1.$$

This would give us the inequality

$$a_1 + a_3 + b_3 + b_2 + c_1 + c_3 \leq 0.$$

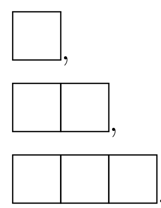
## 3 rank $E = 1$

This is the case where we're looking at maps $f : \mathbb{P}^1 \to Gr(1,4) = \mathbb{P}^3$. Lets write down the quantum cohomology table here as well. These are $1 \times 3$ boxes, so the partitions are

$\lambda = 1$

2

3



Now obviously, $Gr(1,4) \simeq Gr(3,4) \simeq \mathbb{P}^1$, so we should expect to see the same table. But, let's check the computations out explicitly anyways to see what's up.

| | $\sigma_0 = 1$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
|---|---|---|---|---|
| $\sigma_0 = 1$ | 1 | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
| $\sigma_1$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $q$ |
| $\sigma_2$ | $\sigma_2$ | $\sigma_3$ | $q$ | $q\sigma_1$ |
| $\sigma_3$ | $\sigma_3$ | $q$ | $q\sigma_1$ | $q\sigma_2$ |

In this case, to convert to subsets $I \in [n]$ from $\lambda$, our formula becomes simply $I = 1 + \lambda_{1+1-1} = 1 + \lambda_1$. So doing the conversion table, we would get

$$\begin{aligned} \lambda = 0 &\rightsquigarrow \{1\}, \\ 1 &\rightsquigarrow \{2\}, \\ 2 &\rightsquigarrow \{3\}, \\ 3 &\rightsquigarrow \{4\}. \end{aligned}$$

So here, for a degree 1 inequality, we can do something like $\sigma_1 \cdot \sigma_3 = q\sigma_0$. This translates to

$$\langle \sigma_{\{2\}}(F_\bullet), \sigma_{\{4\}}(G_\bullet), \sigma_{\{4\}}(H_\bullet) \rangle_1 = 1,$$

which translates to

$$a_3 + b_1 + c_1 \leq 1.$$

For a degree 0 inequality, we can try $\sigma_2 \cdot \sigma_1 = \sigma_3$. Here we get

$$\langle \sigma_{\{3\}}(F_\bullet), \sigma_{\{2\}}(G_\bullet), \sigma_{\{1\}}(H_\bullet) \rangle_0 = 1.$$

Then here we get the inequality

$$a_2 + b_3 + c_4 \leq 0.$$